RETI SAPERI LINGUAGGI

# Numbers in mind.
# Between intuitionism and cognitive science

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**Abstract**

This paper focuses on Dehaene's view of the core abilities underlying human numerical cognition, and considers whether it lends support, as it is claimed by Dehaene, to the intuitionist school in the tradition of Brouwer. The conclusion is that an intuitionist, Brouwerian perspective is in principle incompatible with second-generation embodied cognitive science, which instead follows Dehaene's approach.

**Keywords**

numerical cognition, Brouwer's intuitionism, cognitive science

## Introduction

In considering the issues that currently polarise the philosophy of mathematics, it is reasonable to conclude that the questions raised by foundational schools are only apparently out of fashion. All different points of view (formalism, logicism, intuitionism) represent still today important reference points that must be kept in mind when it comes to framing new research in the philosophy of mathematics. According to some authors, the main question of the philosophy of mathematics was and still is one related to the "foundations of mathematics" in the "three senses of foundation: metaphysical, epistemological and mathematical" (Shapiro, 2004, p. 37). This is true even despite the fact that the large and ambitious foundational school programs which begun in the first half of the twentieth century and the less ambitious programs of the second half of the twentieth century (which turned out to be simple variations on the themes raised by Frege, Hilbert and Brouwer) were not able to fully address the issues regarding the nature of mathematics and to find a foundational and definitive aspect or conclusion. As a matter of fact, all this led to the opposite outcome: the conclusion that nothing similar actually exists.

Considering all this, several authors believe that the time has finally come to rethink the philosophy of mathematics, moving away from foundational approaches, creating a philosophy of mathematics that is no longer independent from and which can be part of a general philosophy where several questions might be properly addressed. Bonnie Gold (1994) is one of the supporters of this idea and in What is the philosophy of mathematics and what should it be? she listed 38 questions concerning the main mathematical issues that should be investigated in order to gain interesting insights in fields that go beyond the realm of mathematics. Many of these issues touched on traditional questions of the philosophy of mathematics (e.g. the questions raised about the existence of mathematical entities and its certainty, the role of mathematical demonstration, the limitations of intuition or other forms of knowledge, etc.). Other questions were instead aimed at investigating the nature of the science of mathematics, the relationship existing between the first concrete experiences and the most developed mathematical knowledge, or the relationship between mathematics and natural sciences. Indeed, over the last years, these questions have gone beyond the pure realm of philosophy and they have become the object of concrete scientific research in the field of cognitive sciences and, more specifically, in the field of cognitive psychology. Considering the amazing progress made recently by cognitive sciences, it was natural to expect that neurobiologists and psychologists would get more interested in the field of mathematics, particularly as they started looking for the foundations of mathematics in human cognition.

Nevertheless, a particular and interesting phenomenon emerged: instead of deepening the analysis of the cognitive and cerebral mechanisms that are at the basis of mathematical constructions, some cognitive scientists (in fact, only few) decided to side with one of the foundational schools created in the twentieth century, because of the internal dynamics of their own theoretical/experimental approach. A clear example of this phenomenon is represented by the French neuroscientist Stanislas Dehaene. When confronted directly with foundational schools, Dehaene decided resolutely to side with intuitionism. In one of his works, he stated:

> Among the available theories on the nature of mathematics, intuitionism seems to me to provide the best account of the relations between arithmetic and the human brain. The discoveries of the last few years in the psychology of arithmetic have brought new arguments to support the intuitionist view that neither Kant nor Poincaré could have known. These empirical results tend to confirm Poincaré's postulate that number belongs to the "natural objects of thought," the innate categories according to which we apprehend the world (Dehaene, 2011: 226-27].

After dismissing other alternatives (platonism, formalism, logicism), Dehaene opted for intuitionism, because in his view it succeeded in clarifying the relationship between arithmetic and the organisation of the brain. This point is made even clearer in a document that Dehaene prepared in 2006 as introduction to his course, in which he explained:

> The position I am defending (in this course) and which you can qualify as intuitionist, does not belong to any of these fields [Platonist and formalist]. It postulates that the cognitive foundations of mathematics must be sought in a series of fundamental intuitions of space, time, and number shared by many species of animals and which originate in a distant past where these intuitions played an essential role to survive. Mathematics is built on the formalisation and creation of a conscious relationship among these different intuitions. This position is close, but not identical, to the mathematical intuitionism of Brouwer and Poincaré. The difficulty lies in precisely defining what is meant by intuition. It is not certain, in fact, that the variety of properties that are attributed to it arise from a single cognitive process. Nevertheless, in the domain of elementary numerical cognition, recent research have defined very precisely a body of knowledge that can be qualified as 'numerical intuition' or 'number sense' (Dehaene, 2006: 277-278).

Based on the results of his numerous experimental studies, Dehaene concludes that he has enough arguments to reject platonism and formalism, and to opt for intuitionism. In doing so, he shows a particular interest in the constructivist perspective of intuitionism, proving that foundational approaches are often popular also among non-philosophers. This paper tries to highlight the reasons that led Dehaene to support intuitionism and evaluates how much this decision was justified.

# 1. Experiments on numerical cognition

In 1997 Stanislas Dehaene published the first edition of The Number Sense. In his book, Dehaene hypothesises that human beings are born with a "number sense" that they share with other animals and that this instinct is the expression of the functioning of a "mental organ", a set of brain circuits that exist also in other species. According to Dehaene (1997), this "mental organ" works as an accumulator, namely a kind of approximate counting device that allows us to perceive, store, and compare numerical quantities. To better understand these dynamics, it is useful to recall the metaphor of the water tank used by the author. According to the French neuroscientist, we should imagine each entity that must be counted as a specific quantity of water that is added to a tank; by marking the water-level of the tank, it is possible to compare sets of different sizes. Similarly, it is also possible to perform operations of addition and subtraction by simply adding or removing a specific quantity of water. The accumulator would work, therefore, by recording different events and by representing each event with a "drop of water". In this way, different numbers would be represented by different levels of water; however, because such a system fails to represent the exact level of water, the functioning of this mechanism is affected by two different effects: the distance effect and the size effect. Because of the distance effect, the difficulty in distinguishing the difference between two sets is higher, the lower the difference between them. Similarly, because of the size effect, the difficulty in distinguishing two sets increases as their sizes do. The accumulator cannot therefore be accurate in these cases, because it is not able to represent the exact level of water. As a series of experimental data proved, this ability is not symbol- or language-dependent, it allows to approximately recognise quantities, and it is shared by other animals (pigeons, mice, crows, non-human primates) and infants1.

The American researcher Karen Wynn (1992) provided the clearest demonstration to the fact that even children are endowed with this inbuilt "number sense" and that they are able to perform simple operations of addition and subtraction. She devised a series of brilliant experiments that exploited the children's ability to recognise physically impossible events by observing them over a long period of time2. In these experiments, five-month-old children were placed in front of a puppet theater equipped with a screen that could move up and down. Initially, the theater was empty and the researcher introduced one puppet, placing it on the stage. Then, the researcher moved the screen up in order to hide the scene and introduced a second puppet. At this point, the screen was lowered, so that the children could see two puppets on stage. The sequence was then repeated several times, but in some cases the puppets shown to the children represented impossible results (for example, 1 + 1 = 3 or 1 + 1 = 1). In these cases, the children watched the scene longer compared to what they did when two puppets appeared. Wynn obtained the same results even by modifying the experimental procedure in order to test the ability of children to understand subtraction. In this experiment, an exact number of objects guided the behavior of children, not a rough distinction (for example, only one puppet vs. many puppets). In both conditions tested, Wynn noticed that at the end of the experiment, the children paid more attention when the final result showed an inconsistency in numbers.

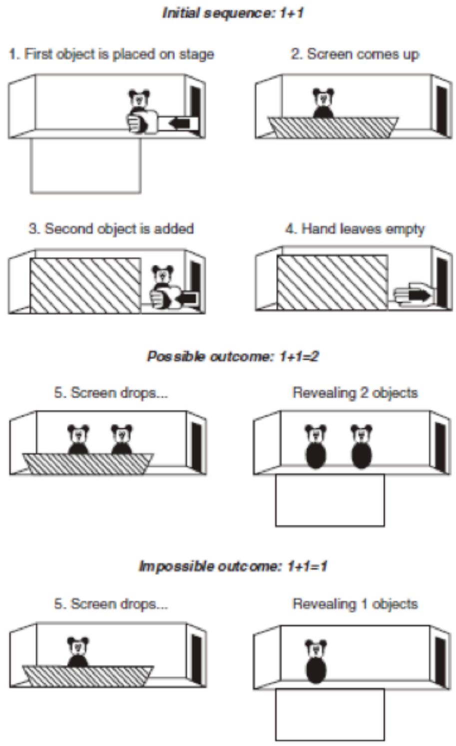


Fig 1. The experimental setting used by Karen Wynn

As mentioned above, Karen Wynn's experiments involved only 5-month-old children but they were repeated also on infants that were just a few days old (Baillargeon et al.,1985), hence providing sound evidence to the hypothesis that the "number sense" is an important part of the heritage of our evolution history. Besides, Dehaene considers it as the seed that allows for the development of mathematical skills. Nevertheless, despite the fact that human adults use a symbol-system in order to represent and work with numbers, some experiments showed that they keep exploiting their analogical, non-verbal representations of numerosity when facing tasks that are similar to those involving the children of Wynn's experiments[3]. Furthermore, other data that indicate the existence of a linguistically independent "number sense" in adults are provided by several neuropsychological studies performed on patients suffering of "dyscalculia"[4], which point to a separation between the system specialised in processing numerical information and the other semantic mechanisms, highlighting therefore the fact that arithmetical skills (even the most accurate and therefore not only those that provide approximations) are not only linguistically independent, but they are independent also of other skills, visual and spatial skills in primis (Butterworth, 1999; Dehaene, 2011).

To sum up the view of the French neuroscientist, Dehaene affirms that there are two different cognitive systems relating to mathematical skills:

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1 All these experiments question the hypothesis put forward by Piaget (1952), in which he stated that there was an indivisible relation between the structures of general intelligence and the evolution of numerical skills.

2 Several previous experiments had shown that children were astonished when they experienced unexpected events that didn't respect the basic laws of physics. These surprising events caused children to watch the scene for a longer time (for further information, cf. Wang & Baillargeon, 2008).

3 A first experiment was performed in 1974 by Buckley and Gillman (1974), who asked some adults to compare two different groups of points. Regardless of the way in which the inputs were presented, the two researchers noticed two complementary effects: first of all, errors increased according to the reduction of the difference between the sizes of the two groups (distance effect); secondly, the errors increased when the numerosities to compare were very small (size effect). The same result was achieved years later by Van Oeffelen and Vos (1982). Similar results were obtained also by other studies (Barth et al., 2005).

4 Deficit in the ability of manipulating numbers that might affect both children and adults

The first system is not based on symbols and it is approximative; it is based on the estimation of quantities; and it involves both a simple process of comparison and a series of basic arithmetical operations like addition and subtraction.

The second system is based on symbols and it is language- and culture-dependent; it is typical of adults; and it is founded on the ability of counting, therefore on a numerical system and on all arithmetical operations.

The first system is the accumulator; it is independent of culture and language and it is made possible by a part of the brain responsible for the perception and representation of numerical quantities. The characteristics of this system link it to the proto-arithmetical skills of infants and animals. On the other hand, the second system is culture-dependent and it depends also on a learning process of symbols and rules; therefore, it is closely linked to language and hence typical of human adults.

According to Dehaene, the awareness of the cultural nature of exact arithmetic is due to the "courage" and talent of some anthropologists and linguists "who took the pains to travel great distances in order to investigate the mathematical competence of remote cultures" (Dehaene, 2011: 260) in which there is a minimal mathematical vocabulary that in most cases only includes the words for "one", "two", "three", "a lot". In particular, the most important study investigating the numerical cognitive abilities of "primitive" populations was carried out by Pierre Pica and his colleagues (2004), following a visit paid by the author to the Munduruku tribe, an indigenous people living in the autonomous territory of the Para state (Brazil) and speaking a language that belongs to the Tupi family. In their language, the Munduruku have names only for the numbers going from 1 to 5. Despite this fact, according to Pica, the Munduruku can identify quantities over five and they are aware of the basic laws that regulate the development of the cardinality of sets (union and addition, separation and subtraction, order of quantities). On the other hand, they face greater difficulties when dealing with arithmetical tasks that require an exact result (Pica et al., 2004; Dehaene et al., 2006; Dehaene et al., 2008; Frank, 2008). Therefore, despite the fact that these studies stressed the importance of a verbal counting system, they highlighted also that language - and culture – is not always involved in the manipulation of numbers and consequently human beings (adults and children alike) can exploit different procedures in order to establish the quantity of objects in a specific set.

According to cognitivists, the quantification process takes place in three different ways, using estimation, counting and subitizing.

Estimation is used when it is necessary to process numerosities in an undefined way, for example when it is necessary to make an estimation about the number of people in a room (Dehaene, 1997). The approximate number obtained through this operation is always quite accurate, even if the final result may be affected by different factors[5]. Besides, as mentioned in the previous paragraphs, estimations are affected by the distance effect and the size effect. In other words, it is easier to recognise the difference between 80 and 100 compared to the difference between 81 and 82. Furthermore, when the distance is the same, it is easier to recognise the difference between two small numbers (for example 10 and 20) compared to the same difference between two big numbers (for example 90 and 100). Compared to estimation, counting gives more accurate results and the psychologists Gelman and Gallistel (1978) identified its five main principles:

One-to-one correlation: each element counted is identified only by one single number;

stable order: numbers must be ordered in a reproducible sequence;

cardinality: the last number of the series represents the feature of the entire set;

abstraction: all sets of entities can be counted (objects, events, mental constructions);

irrelevance of order: the order in which the different elements that must be counted are processed is irrelevant to their counting.

Finally, the third process mentioned above represents still today one of the most debated and controversial issues. Subitizing is a term derived from the Latin word subitus and it refers to a process of rapid and accurate recognition of the numerosity of sets containing a maximum of 4-6 units. The studies on subitizing started right after Mandler and Shebo (1982) published the results of an experiment in which they asked some volunteers to determine, as fast as possible, the number of items shown on a screen. The results proved (figure 2) that the reaction times recorded a linear increase of 200 ms only for sets consisting of 4 to 6 items, while for sets consisting of 1 to 3 items the reaction times were much faster and they increased only slightly according to the variation in the number of items. Finally, for sets consisting of more than 7 items, the reaction times were more or less stable.

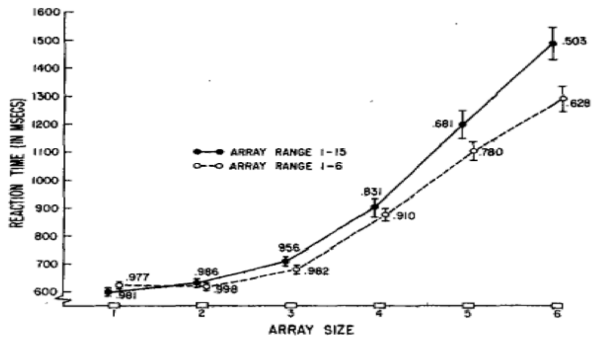



Fig 2. Reaction times in identifying the number of items presented for a time of 200ms (Mandler & Shebo, 1982)

The research proved that for quantities from 1 to 3 or from 1 to 4, the volunteers did not count the elements one by one, but they recognised immediately the total number of elements. In order to provide an explanation to these experimental results, the two researchers put forward the hypothesis that subitizing was due to the immediate perception of spatial configurations, according to which the quantity one must be represented by one point, two points necessarily form a line, three is identified immediately as a triangular configuration, while four is "subitized" only when it can be displayed in a canonical structure as a square or a triangle with a point in the center. For numbers over four, the variability of configurations increases, thereby making immediate recognition impossible. Authors like Gallistel and Gelman (1992) support instead the idea that "subitizing" is nothing more than a very rapid enumeration that exploits non-verbal labels, in other words a form of pre-verbal and innate counting ability. Different explanations were proposed by Alan Leslie and his colleagues (1998), and by Tony Simon (1999). Their explanations were inspired by the theory of "attention indicators" (object files). According to this theory, a person identifies an object because an "indicator" allows him to follow objects as they move by linking together the different perceptions of the same object that were recorded and distributed over time and space. The num-

5 For example, humans tend to overestimate the number of objects if they are evenly spread, while at the same time they tend to underestimate objects when they are unevenly scattered around (cf. Dehaene, 2011; Frith & Frith, 1972).

ber of simultaneously available indicators, however, is limited to four, and this explains why it is impossible to simultaneously follow more than four objects in the same moving group. Starting from the indicators, it is therefore possible to infer the numerosity of the sets of objects (for example, two sets of objects can be compared by making a correspondence term based on term-level indicators) and do some simple operations. Starting from these ideas, Stanislas Dehaene suggested the existence of two basic cognitive systems (core systems), in which the first one (accumulator) allows to make approximate representations of numerosities featuring an arbitrary number of objects, while the second one (subitizing) allows accurate representations of very small numbers of objects, something that from a phylogenetic and ontogenetic point of view give existence to formal arithmetic. Dehaene states that:

> The current consensus is that we have not just one, but two systems for representing a number of objects without counting. The small-number system, sometimes called the "object tracking" system, only represents sets of 1, 2, or 3 items. It lets us track their trajectories quite precisely, and therefore gives us an exact mental model of what happens when one object moves in or out of a small set. The approximation system, on the other hand, can represent any number, large or small. It allows us to compare them or to combine them into approximate operations (Dehaene, 2011: 258).

He continues arguing that:

> So how does subitizing work? Current research suggests that we have 3 or 4 memory slots where we can temporarily stock a pointer to virtually any mental representation. This memory store is called 'working memory' — a transient supply that keeps the objects of thought on-line for a brief moment (Dehaene, 2011: 259).

In other words, according to Dehaene, when we are dealing with a number of objects under three, the working memory has enough "slots" that allow for an exact estimation. When there are more than three or four items, though, the second system starts functioning, but unlike the system of "object files" it is no longer accurate, because it treats these objects as "noises" and therefore "seven and eight overlap, while two and eight do so far less" (Dehaene, 2011: 260).

As mentioned in the previous paragraphs, the debate on how the process of subitizing actually works is still in progress, but what seems to be accepted is the existence of an innate ability in children to represent the transformation of a set (addition and subtraction) and to understand the relationship between two numbers, even before the development of a verbal counting system that manifests itself through the representation of the cardinal value of a set. These experimental data have indeed shown that children (as well as some animals) are equipped, even at an early age, with a series of preverbal numerical skills that allow them to understand some events taking place in their environment. This does not mean that children always exploit this ability, nor that they are endowed with an infallible numerical competence. This is easily explained by the fact that children (like adults) exploit more easily perceptive indices or more random - but cognitively less expensive - heuristic experiences. However, the question that unfortunately remains still unanswered concerns understanding how children are able to apply accurate numerical representations starting from approximate non-verbal representations. In other words, although children at this stage have already access to representations of approximate numerical quantity, they are unaware of it. Now, it seems clear that the concept of number must be accessible to consciousness, since the children who have acquired it are able to use numbers to distinguish elements in collections of objects. Therefore, to explain the acquisition of mathematical concepts, we must answer the two following questions. How can the concepts of approximate numerosity become an object of thought that is so accessible to our consciousness? How are these concepts refined and specified in such a way as to become numbers? Unfortunately, starting from these experimental results, there is currently no model that can truly demonstrate the role of language in the development of numerical skills starting from approximate pre-verbal skills[6].

## 2. Intuitionism and the impact of Dehaene's findings

As stated at the beginning of this paper, Dehaene is one of the few cognitive scientists who took a clear position in the debate between foundational schools. Considering recent neuroscientific discoveries, he sees the intuitionism of Poincaré and Brouwer as the most correct theory and he devotes entire pages of his most famous book, The Number Sense, to Poincaré, who claimed to be able to intuitively determine the certainty of a mathematical result even if proving it sometimes required several hours of calculation. Besides, Poincaré claimed that the basis of the works of mathematics could be perceived by spatial, motor or numerical intuition. Dehaene, therefore, supports Poincaré's intuitionism. He supports also his idea that the methods used to teach and learn mathematics should necessarily make use of intuition and reasoning by analogy; he agrees with Poincaré on the fact that intuition and reasoning are equally essential in order to create new facts in mathematics. However, what seems to convince Dehaene about the soundness of Poincaré's thought is an idea that Hadamard presents in his The psychology of invention in the mathematical field, referring to a passage of Poincaré's work where he talks about mathematical discoveries and affirms that "what's most striking at first is this appearance of sudden illumination, a manifest sign of long, unconscious prior work. The role of this unconscious work in mathematical invention appears to me incontestable" (Hadamard, 1945: 14). This consideration seems to convince Dehaene even more about the fact that mathematicians, at least at the beginning of their studies, "have claimed to possess a direct perception of mathematical relations. They say that in their most creative moments, which some describe as "illuminations", they do not reason voluntarily, nor think in words, nor perform long formal calculations" (Dehaene, 2011: 136). Therefore, these ideas follow a tradition in which intuition is seen as a source of direct, immediate knowledge, unaffected by inferential mediation, and which will be harshly criticised by Dieudonné, who stated that "the intuition of the whole is a great mystification, because no one I know has insight in the true sense of intuition, that is, immediate knowledge of whole numbers greater than ten. Consequently, to say that you have intuition of integers greater than ten is a big fraud" (Dieudonné, 1981: 23). Dehaene seems to escape Dieudonné's criticism because, while accepting the idea that intuition is immediate, direct, non-linguistic knowledge, he compares this concept of intuition to his idea of subitizing which, as we have seen, is only valid for the first three positive integers and that after three proves to be fallible and subject to the distance effect and size effect.

Nevertheless, it is useful to remember that Poincaré provides different meanings to the term "intuition". For example, he writes the following:

> We have then many kinds of intuition; first, the appeal to the senses and the imagination; next, generalization by induction, copied, so to speak, from the procedures of the experimental sciences; finally, we have the intuition of pure number, whence arose the second of the axioms just enunciated, which is able to create the real mathematical reasoning. (Poincaré, 1907: 20).

6 For a full review on why it is not possible to give for granted the relation between elementary skills and approximation skills, cf. Rips *et al.*, 2008.

According to Poincaré, the last kind of intuition mentioned is the only one that can provide certainty, because it is the only one that is a clear manifestation of a property of intelligence itself. Therefore, when Dehaene affirms of supporting the intuitionism of Poincaré and simplifies it to the concept of subitizing, he does so without considering that for Poincaré it is also a tool for demonstration and therefore not only a number generator. In other words, Poincare's intuition is a broader concept compared to "subitizing" as defined by Dehaene. In fact, even if Poincaré has the perspective of an intuitionist, his philosophical thought undoubtedly integrates elements of formalisation. Taking a look to Poincaré's famous passage "Thus logic and intuition have each their necessary role. Each is indispensable. Logic, which alone can give certainty, is the instrument of demonstration; intuition is the instrument of invention" (Poincaré, 1907: 23), it is possible to notice that it does only summarise a very important argument of his essay The value of science, in which he dwells upon the dialectic existing between sensitive intuition and analytical procedures, which Poincaré calls verifications and which are based on syllogism, replacement and nominal definition. It was precisely because of these remarks that the positions of Poincaré were defined as belonging to a kind of semi- or pre-intuitionism.

Luitzen Egbertus Jan Brouwer is considered one of the greatest mathematicians of the twentieth century. He discussed the relationship existing between intuitionism and the thought of Poincaré, highlighting the confusion that Poincaré allegedly made between "the language of mathematics and the real mathematical construction" (Brouwer, 1907: 176). While both Brouwer and Poincaré share the belief that intuition is the only thing that can guarantee the certainty of mathematics, Poincaré does not make a clear distinction between language and mathematics. Contrary to Poincaré, Brouwer considers that intuition is the only basis of mathematical construction. Moreover, Brouwer considers formalism as useless and even harmful as it promotes the diffusion of "paradoxes" (Brouwer, 1907: 176).

If we want to take a closer look to Brouwer's thought, we should start by considering that during his lifetime, Brouwer always showed an interest in Eastern mysticism and in the belief of the duality of mind-body. In his writings, it is easy to find references to concepts such as karma, the life of the soul, reincarnation, and immortality. Besides, the same references can be found also in his writings on mathematics. Although it is easy to understand why members of the Western mathematics community sometimes did not see the relevance of these proposals, it is important to highlight that Brouwer's mystical beliefs are at the heart of his philosophy. For Brouwer, mysticism is indeed a source of knowledge just as reason is. Therefore, it is not possible to completely dissociate his mystical and moral thesis from his mathematical thesis without significantly altering the latter, especially with regard to the key element of its constructions, namely, "Primordial Intuition". When dealing with primordial intuition (hereinafter PI), the subject becomes aware of the discrete elements in time, in other words he becomes aware of two discrete entities, one belonging to the present and the other belonging to the past. In this way, as it is described by Brouwer, intellect takes shape: "By a move of time a present sensation gives way to another present sensation in such a way that consciousness retains the former one as past sensation" (Brouwer, 1948: 1235). In other words, when PI occurs, the consciousness retains two neighbouring elements that are different but unified. The two elements are not identical, but they do form a single unit that Brouwer calls "twoity". It is important to highlight that when dealing with PI, individuals are affected by the action of consciousness, which is described by Brouwer in different ways, not so much as an event but rather as a phenomenon or a process.

Primordial intuition can be endlessly repeated; it depends only on the free will of the subject, thereby producing sequences of increasingly complex mental objects simply as repetition of the primordial act. With a twoity, it is possible to build a threeity; with a threeity it is possible to build another construction, and so on. Brouwer calls these sequences obtained through the repetition of PI with the name "causal sequences", because of the organisation of these perceptions into strings of constructions linked by cause-effect relations. Since PI is responsible for the mental construction of all entities, including mathematical ones, all the relations that it builds are characterised by causality.

Following what has been outlined in the previous paragraphs, it is clear how it is possible to build constructions of mathematical entities starting from PI. When a subject builds the first twoity, it can be considered separately from another moment in time, becoming in this way a new "entry" for the same phenomenon that leads to the construction of a threeity.

The before-after relation contained in the twoity does not indicate only the presence of two elements, but it represents also the beginning of another sequence that will produce another element. By abstraction from their temporal content, these sequences of n-units will become the sequence of ordinal numbers and subsequently also the whole of mathematics. Therefore, according to Brouwer, all numbers - ordinals, natural and other - are constructions obtained from reiterations of PI: "This intuition of two-oneness, this ur-intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers" (Brouwer, 1912: 12). Indeed, it is always possible for a person to build a higher number, because the number of possible repetitions of PI is unlimited; the only limit is represented by the free will of the creator.

Therefore, according to Brouwer, there is a distinction in terms of the activity of consciousness between the naive stage (the initial stage of consciousness, primordial attention, PI) and the causal stage. During the latter, PI is activated by unifying separate precepts in order to build complex mental objects. In other words, in the initial stage, the intellect (the mind) is at rest, it passively receives sensations, it is spontaneous and instinctive. The same cannot be said during the causal stage, because intellect is activated by sensations and it starts building sequences and creating relationships between them. In the third stage, the social stage, it is possible to make use of the causal sequences created by other individuals in order to build more complex causal sequences in networks of social interactions. The fruits of such cooperation are the formulation of hypotheses and science.

It is in this stage that language becomes useful, because it simplifies the transmission from one subject to the other. It also plays an important role in helping memory, though memory is far from infallible: "The role of mathematical language can only be that of an aid to help remember mathematical constructions or construction methods or to suggest them to others, sufficient for most practical purposes but never completely safeguarding against error" (Brouwer, 1947: 339). Therefore, according to Brouwer, language does not play a role in the construction of mathematical concepts. In fact, mathematics is a free mental construction based on PI and therefore it precedes all forms of linguistic description. Language can only describe the constructions made by PI. This separation of the language of mathematics from mathematics is the subject of the First Act Of Intuitionism:

> Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the

> twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics. (Brouwer, 1981: 4–5)

As for logic, Brouwer describes it as a formal and sterile language without any constructive power, obtained through the simple observation of the form of linguistic descriptions of mathematical processes. For Brouwer, mathematical entities are therefore mental constructions originated by the fact that subjects are able to distinguish between the content of sensations and the emergence of the moment when they occur. It is in fact in this stage that causal attention starts. At this stage, mathematical attention is paid to the world of sensations, thereby contributing to the construction of a world of perceived objects, something that Brouwer calls "the world outside of the subject" (Brouwer, 1948).

Starting from these brief remarks on Brouwerian intuitionism, it is possible to say that Dehaene differs from Poincarè, while he seems to agree more with Brouwer, at least considering their starting points. Both Brouwer and Dehaene agree on the fact that mathematics is independent of language and logic, and that mathematics is entirely man-made, or better said it is entirely made by man's free mind. Furthermore, it might be interesting to highlight that there are several similarities between the continuous "entry" of Brouwer and the supramodal representation mentioned by Dehaene. As a matter of fact, both authors support the idea of a system that allows individuals to build discrete objects and to represent the difference of a limited number of objects at the same time7. Besides, exactly as an "entry", the analogical representation of a numerical quantity works independently of traditional senses like touch, sight, etc. In Brouwer, the entry helps in distinguishing two elements by placing them at the same level, starting from a temporal level and then, gradually, bringing them to an abstract, numerical level. In the same way, also the supramodal representation of numerosity leads to a specialisation: as mentioned in the previous paragraphs, Dehaene supports the idea that starting from some approximative representations of numerosities, it is possible to obtain accurate numbers that may be used in order to perform calculations or other arithmetical operations. Therefore, in both cases these representations allow to insert an unlimited number of new elements8. Besides, as the "entry" of Brouwer allows to recognise not only mathematical entities but also other kinds of entities, Dehaene contends that the neurones involved in the supramodal representation of numerosity might play a role also in non-mathematical constructions (for example, for other continuous quantities like lightness or format, cf. Pinel et al., 2004).

However, the language-less feature of mathematics was considered by Brouwer as a necessary condition to practice it without violating his mystical conception. For Dehaene, the language-less feature becomes the fundamental basis of elementary, approximative mathematics, shared by humans and other animals. While both authors share the need of a conception of mathematics not based on language, for Brouwer this need involves the entire realm of mathematics because of a series of purely human reasons (mystical needs), while for Dehaene it involves only approximate mathematics, which is accessible to animals and to populations who do not have many names for numbers. Therefore, the mathematical notion of Brouwer is much more general than the one described by Dehaene, as it includes all mental construction processes, conscious and unconscious, because according to Brouwer all our mental skills are based on PI. For example, Brouwer thinks that the construction of sensorial objects is possible thanks to mathematical skills. As Ewald rightly affirms:

> [...]the basic intuition of two-oneness underlies, not only pure mathematics and theoretical logic, but also many scarcely conscious everyday mental processes, including the mental organization of the objects of the external world; so that in this sense mathematics is broader than any of the special sciences, and does not rest upon any foundation more fundamental than itself (Ewald, 1996: 1173).

On the other hand, according to Dehaene, the term "mathematics" has a narrower connotation that includes only the processes and representations involved in tasks like counting, addition, multiplication, comparison of numbers, etc. In other words, Dehaene's conception of mathematics is closer to the concept of common sense. Considering this, it is fairly easy to dissociate the pure mathematics of Brouwer, which corresponds to the formal mathematics of Dehaene, from general mathematical skills, which are instead the equivalent of general cognitive abilities. Ultimately, what is built purely by PI without any help of the other senses, according to Brouwer, corresponds to the conception of common sense of mathematics as described by Dehaene. As a matter of fact, according to Dehaene, exact mathematics is symbolism, therefore a type of language. In other words, it is what in the mystical view of Brouwer was considered as something to avoid because it was linked to activities aimed at the outside world. Contrary to what Dehaene contends, Brouwer supports the idea that language cannot guarantee mathematical exactness. According to Brouwer, the truth and validity of mathematics must be found in the mental process. Brouwer's acrimony against everything outside the mind, a hostility due to the mystical definition of the "search of happiness", will lead him to convey a secondary role to the body, creating another salient point on which Dehaene and Brouwer will substantially differ. Specifically, Dehaene attributes a fundamental role to the embodied component of mathematical knowledge because he is convinced of the soundness of the data obtained through studies carried out on non-Western populations with minimal mathematical vocabulary that in most cases only includes words for "one", "two", "three". Based on the outcomes of these studies, Dehaene believes that the transition towards counting systems with numbers over "three" required these populations to count by using different parts of the body. This is the case, for example, of the Warlpiris (Australia), who use the terms for "one" and "two" in order to create different combinations and count sets containing up to four elements (Dehaene, 1997). The same happens in certain tribes of the Torres Strait islands, in the northern part of Australia, where natives use the words urapun and okosa for "one" and "two", while the terms okosa-urapun (i.e. 2 + 1) and urapun-urapun (2 +2) are used for "three" and "four". In addition to that, they also have the word ras, which means "a lot" (Ifrah, 1994). The same features can be found among some African people such as the Bushmen (Southern part of Africa), the Zulus and the Pygmies (Central part of Africa); some tribes of South America, like the Botucodus (Brazil); and the Vedda people of Sri Lanka.

According to Dehaene, the use of fingers, the entire hand and in some cases the whole body allows some people and tribes with a poor numbering system to designate different quantities and numerosities. An example of this is the Kilenge people in

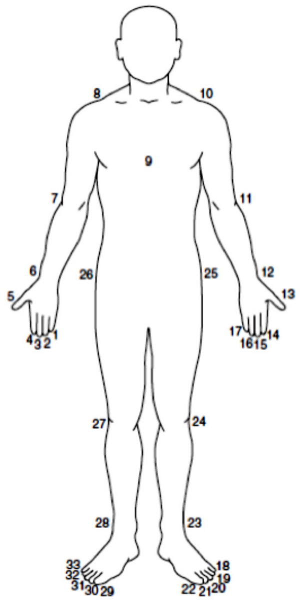
7 According to Brouwer, continuum and discrete are not opposite concepts, but instead they are complementary to each other: each and every one of them is used to define and understand the other. They are essential to each other in order to create twoity: to witness change, it is necessary to witness two different moments in time, before and after. These two moments create a difference in time in which change is observed (Brouwer, 1912 as mentioned in Benaceraff & Putnam, 1964, p. 69).

8 Brouwer's 'entry' refers to a similar idea of specialisation, since it is an endless divisible flow in which natural, ordinal, and real numbers can be defined.

Papua New Guinea. They use the term "hand" to indicate 5, "two hands" for 10, "two hands" and "one foot" for 15, and the term "man" to indicate the number 20. By combining these terms with distinctive words for the numbers that go from one to four, the Kilenge people can count over 20 (35 = "one man, two hands and one foot"). Several aboriginal groups living on the highlands of Papua New Guinea use counting systems that designate different parts of the body in a specific order. For example, the indigenous people living on the Murray Islands in the Torres Strait (Dehaene, 1997) start counting by indicating the little finger of the right hand (1) and then move towards the thumb (5). Then, they move on to the wrist (6), the elbow (7), the shoulder (8) and the torso (9). At this point, they go to the left arm and they follow the same procedure but in reverse order. When they reach the little finger of the left hand (17), they keep counting following a similar procedure starting from the little toe of their left foot (18) until reaching the little toe of the right foot (33), passing through their ankles (23, 28), knees (24, 27) and hips (25, 26). Similarly, the Yupno people of Papua New Guinea can count by indicating different parts of the body.

Fig 3. Body-part counting system.

It is clear, therefore, that despite Brouwer and Dehaene share some common ideas, the intuitionist perspective of Brouwer is in principle incompatible with the second-generation embodied cognitive science of Dehaene. The embodied paradigm represents indeed a turning point compared to the traditional




approach of first-generation cognitive science, since it does not consider anymore the human mind as a simple processor of symbols and of mathematical calculations. Thanks to this new conception, the studies on cognition and learning changed perspective, going from a point of view focusing on the abstract aspects of thought that were governed by formal and culture-independent rules, to a point of view where the mind is context-aware, distributed, action-oriented, holistic, culture-dependent, deeply linked to the principles of biological nature. The crucial change is represented by the fact that intelligent behavior starts being considered as a manifestation of biological bodies that act in their material and cultural environment while at the same time changing it9. All research carried out in the field of "embodied cognition" followed these principles and over the years this approach has clearly shown its potential, demonstrating that the mind must be extended from an abstract place in the head to the actual functioning structure of the brain and to the entire body, until understanding the shape that the mind acquires in the social relationships that it creates and by which it could be affected.

## Conclusion

According to the theories and ideas outlined in the previous paragraphs, it results clear that Dehaene's position - which he describes "intuitionist" - in reality is very different from the traditional philosophical ideas developed by intuitionists. The hypotheses of Dehaene did not only shed some light on the existence of a kind of natural, innate, and biologically based mathematics, but they focused in particular on the concept of numerosity, a term used to indicate the number sense and in particular the sense that recognises the size of a set (which, as we have seen, is subject to the distance effect and the size effect). The hypotheses of Dehaene therefore seem to focus less on the concept of number, because learning the concept of number requires a word or a symbol to represent it.

Of course, not all mathematics has to do with numbers. Beyond the linguistic and conceptual aspects of mathematics, it is possible to find that kind of innate and biologically based mathematics that Dehaene describes in such wonderful terms, the kind of mathematics that belongs to a natural process of adaptation to the environment. As Cellucci claims:

> This is not surprising because mathematics, like all knowledge, is part of a natural process of adaptation to the environment. In this process mathematics plays an important role because, in the course of human evolutionary history, a decisive step towards the achievement of higher cognitive ability was the formulation of hypotheses on the bodies in the environment, their location and their number, which has led man to discover new properties of the environment and to move towards more appropriate behaviors and to have the most successful (Cellucci, 2003: 338).

In other words, thanks to the mechanisms of evolution guided by natural selection, nature shaped living beings in such a way that they became able of performing specific actions and a series of natural mathematical calculations in order to grant their own survival. Therefore, both language-based mathematics and natural mathematics are mathematics, the only difference between them lies in how they are carried out. The former is abstract, symbolic, and rule-based. The latter, on the other hand, is non-verbal, innate, and approximate. They are both equally important, but they are both quite different from what Brouwer had in mind.

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9 The keywords of this new approach indeed are: 'situated', 'distributed', 'social', and 'embodied' (Hutchins, 1996; Lave, 1988; Núñez & Lakoff, 2000).

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